Assignment 0

Instructions: This assignment is a pre-test, designed to show you the approximate mathematical ability for this course. If you find this assignment very difficult, you should be aware that the mathematics in this course will be of similar difficulty, and you should prepare appropriately.

Some notation, which you might not be familiar with:

- If S is a set, the notation $x \in S$ means "x is an element of S" (in other words, the set S contains x inside it).
- The term $\mathbb{R}$ refers to the set of real numbers; $\mathbb{R}^{++}$ refers to the strictly positive real numbers, and $\mathbb{R}^{+}$ refers to the real numbers greater than or equal to zero.
- The symbols $\forall$ and $\exists$ mean "for all" and "there exists", respectively, while $|$ means "such that". For example, $\forall x \in S$ means "for all x in S" while $\exists x \in S \mid x > 0$ means "there is an x in S which is greater than zero".
  - This is mostly for shorthand, and to read references or books. I try to avoid this where possible.

Structure: This assignment has 12 questions, for a total of 0 points and 0 bonus points.

Hand-in: This assignment is due the last class prior the date listed on Canvas or online by the deadline. Late assignments will be penalized at a rate of 50% per day.

1. Suppose that $f : \mathbb{R}^{+} \to \mathbb{R}$ is a continuously differentiable function defined by:

$$f(q) = (1000 - q)(q - 50)$$

Find all of the critical values of f and categorize which ones are maxima and minima. Pay attention to the domain of f.

Solution: This is a constrained optimization problem, but it only has a non-negativity condition, so we can treat it as a unconstrained equation then examine the borders if necessary. To find the critical points, we find the zeros of this function, then consider their derivatives.

$$f(q) = 1050q - 50,000 - q^2 \implies f'(q) = 1050 - 2q$$

This has a unique solution at $q = 525$

We can see that $f''(q) < 0$, therefore this is a maximum.

Hopefully you immediately spotted that this was the same expression as a profit maximizing monopolist, and could immediately know that this was a maximum.

The only other point to check (via the Kuhn-Tucker conditions) is the edge of the domain, where $q = 0$. At this point, $f(q) = -50,000$, and must be a minimum.

You can see, however, that the function grows arbitrarily small as $q \to \infty$ and therefore (a) $q = 525$ is a global maximum, and (b) $q = 0$ is a local minimum.

2. Consider the infinite series:

$$S = a + \delta b + \delta^2 c + \delta^3 a + \delta^4 b + \delta^5 c + ...$$

What is S in terms of a, b, c and δ when $\delta \in (0,1)$?

Solution: This question concerns the properties of geometric series; let's begin under the assumption that these series are convergent, then verify *ex post*. We can note that S is the sum of three smaller series, $S = S_1 + S_2 + S_3$:

$S_1 = a(1 + \delta^3 + \delta^6 + ...)$

$S_2 = b(\delta + \delta^4 + \delta^7 + ...)$

$S_3 = c(\delta^2 + \delta^5 + \delta^8 + ...)$

We can then see that $S_1 = aX$, $S_2 = n\delta X$ and $S_3 = c\delta^2 X$ where:

$$X = 1 + \delta^3 + \delta^6 + ... = 1 + \gamma + \gamma^2 + ...$$

where $\gamma = \delta^3$. Since $\delta \in (0,1)$ then $\gamma \in (0,1)$ (since the product of a number less than one is smaller than the original number). Therefore, we know that X is convergent, and therefore all of the other series are convergent as well. What is the limit? The geometric formula tells us that:

$$X = \frac{1}{1-\gamma} = \frac{1}{1-\delta^3}$$

Therefore:

$$S = S_1 + S_2 + S_3 = \frac{a}{1-\delta^3} + \frac{b\delta}{1-\delta^3} + \frac{c\delta^2}{1-\delta^3}$$

3. Solve the following system of linear equations, when $\delta = 2$

$$\begin{aligned} x &= \delta y + 3z + 1 \\ y &= 3x - 4z + 2 \\ z &= 3x + 8y + 3 \end{aligned}$$

Solution:

This is basic matrix algebra, and is easiest to solve using the augmented matrix form:

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ -3 & 1 & 4 & 2 \\ -3 & -8 & 1 & 3 \end{array}\right]$$

You can then use row reduction to eliminate:

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & -5 & -5 & 5 \\ 0 & -14 & -8 & 6 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 6 & -8 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -4/3 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -7/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & -4/3 \end{array}\right]$$

You could also invert the matrix, if you know how to do that, and then solve the problem that way.

4. Suppose you have a function $f : \mathbb{R} \to \mathbb{R}$ which is continuously differentiable. If the constraint function $c : \mathbb{R} \to \mathbb{R}$ is also continuously differentiable, what are the first order conditions for a maximum of $f(x)$ subject to $c(x) = 0$?

Solution:

This is an application of the Langrange multiplier method; we are looking for a solution constrained to the set where $c(x) = 0$. If we define the Langrange multiplier on the constraint to be λ then this is equivalent to optimizing th function:

$\mathcal{L}(x) = f(x) - \lambda c(x)$ with respect to x.

The first order conditions are:

$$f'(x) - \lambda c'(x) = 0$$

$$c(x) = 0$$

You can think of this as saying that at a solution, the tangent to the function and the constraint must be (proportionally) equal (and therefore have zero directional derivative).

5. Find the maximum of the function $F(x, y)$ such that $x, y \geq 0$ and $C(x, y) = 0$, where:

$$F(x, y) = 3x^2 + xy + y^3$$

$$C(x, y) = 2x + y^2$$

Solution:

Both of these functions are continuously differentiable, so we can apply Lagrange's method, then check the boundary conditions.

$$\mathcal{L}(x) = 3x^2 + xy + y^3 - \lambda(2x + y^2)$$

$$\frac{\partial \mathcal{L}}{\partial x} = 6x + y - 2\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = 3y^2 + x - 2y\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = -2x - y^2 = 0$$

Equations 1 and 2 imply that (by eliminating λ):

$$3y^2 + x - 6xy - y^2 = 0$$

Equation 3 implies that $x = \frac{-1}{2}y^2$. Substituting this into the above, we get:

$$3y^2 - \frac{1}{2}y^2 + 3y^3 - y^2 = 0$$

In other words, the only feasible solution is $x = y = 0$ which meets the constraints. However, we still need to verify maximum - checking a few values shows that it very clearly is not.

6. Suppose $U = \frac{c}{1-\delta}$ and $U' = h + \frac{\delta}{1-\delta}d$. If we define $S = h - c$ and $L = c - d$, under what conditions on S and L is $U > U'$?

Solution:

This is mostly just algebra; the expression in question is:

$$\frac{c}{1-\delta} > h + \frac{\delta}{1-\delta}d \iff \frac{c}{1-\delta} - c\frac{1-\delta}{1-\delta} - \frac{\delta}{1-\delta}d > h - c$$

$$\iff \frac{\delta c - \delta d}{1-\delta} > h - c \iff \frac{\delta}{1-\delta}L > S$$

7. Suppose $q \in [0,1]$ maximizes $qA(p) + (1-q)B(p)$ where $p \in [0,1]$ and:

$$A(p) = 3p + 2(1-p)$$

$$B(p) = 2p + 4(1-p)$$

If we define $f(p) \equiv q$, is f a function? Sketch the plot of f in a plot with p on the x-axis and q on the y-axis.

Solution: Notice, for a fixed value of p, only one of three things can be true here:

1. $A(p) = B(p)$ which can occur is and only if $p = 2/3$
2. $A(p) > B(p)$ which can occur is and only if $p > 2/3$
3. $A(p) < B(p)$ which can occur is and only if $p < 2/3$

In Case 2, the maximizer is a unique (corner solution) where $q = 1$. In Case 3, $q = 0$. In Case 1, there is no unique solution: every value of $q \in [0,1]$ is a solution.

Thus, $f(p)$ is not a function. Try graphing it for yourself!

8. If you randomly choose a real number between zero and one, then created a coin which has the drawn number as the probability of coming up heads, then flip it, what is the probability it will come up heads?

Solution:

Let E be the event that the coin lands on heads, and let X be the random number that was chosen between zero and one. We want $P(X)$, but we don't know the value of X. However, *conditional* on knowing X, we know that $P(E|X = x) = x$.

By the Law of Total Probability:

$$P(E) = \int_0^1 P(E|X = x) f_X(x) dx = \int_0^1 x f_X(x) dx = E[X]$$

However, we know that X was chosen randomly between 0 and 1, so $E[X] = \frac{1}{2}$. Therefore, $P(E) = \frac{1}{2}$.

9. Suppose you have 10 cars, and you have 17 students. If you assign students to cars, what is the smallest number of students that can be in the car which is the most full? Explain.

Solution: This concept is known as the Pigeonhole Principle. We want to find the smallest number of students in the most full car. If we start assigning students to cars, we can place 10 students in the 10 cars, with 7 left over.

On the second pass, we can assign 7 students to 7 of the 10 cars. The most-full car will have two students. If we re-arrange any student from a car of size two we will create a pair of cars with either (a) 3 and 0 students, (b) 2 and 2 students, or (c) 2 and 1 students. In no case can we reduce the size of the largest car.

10. Suppose there is a belief about the likelihood of a hypothesis being true which we call p; *ex ante*, the prior probability of the hypothesis being true is $p_0 > 0$. Suppose we perform a sequence of independent tests, $t = 1, 2, \ldots$, each of which returns either T or F, which is correct with probability $q > \frac{1}{2}$.

    (a) What is the posterior probability of the hypothesis being true after t tests, denoted p_t, if the probability of the hypothesis being true after $t - 1$ tests is p_{t-1}.

    (b) How many "positive" tests in a row would it take to reach 95% confidence in the truth of the hypothesis, if you start with $p_0 = 0.5$?

11. Suppose there are a large number of infinitesmally sized firms, indexed by their marginal product of labour, $0 \leq l < \infty$, which is distributed according to the PDF f. Each firm

demands an input, q, based on their marginal product. This implies the total demand, of all firms that buy the product, is:

$$Q = \int_{l=0}^{l=\infty} q(l)f(l)dl$$

(a) Suppose that input demand depends both on l and a technological parameter $\alpha \in \mathbb{R}$; that is, $q(l, \alpha)$ is a smoothly differentiable function on its domain. What is the change in Q with respect to α?

(b) Now, suppose that both q and f depend on α, as above. What is the change in Q with respect to α?

(c) Instead, suppose due to government regulation, all firms with a marginal product of labour less than L drop out of the market. What is the change in Q with respect to L?

12. Consider the family of functions $u_y(x) = -(x - y)^2$. Suppose that $a > b$. Furthermore, suppose that $x_1 < x_2$, and that $u_a(x_1) > u_a(x_2)$. What can you say about $u_b(x_1)$ and $u_b(x_2)$? Be specific.

End of Assignment